

MAS212 Linear Algebra I 2007-2008

Coursework 9

Please hand in your solutions of the **starred** exercises by **2pm on Friday 7 December** using the [blue Linear Algebra I Collection Box](#) on the Ground Floor.

Don't forget to put your **name** and **student number** on your solutions, and to **staple** them.

Exercise* 1. (2 marks) Prove the Pythagorean Theorem in $\mathbb{R}^n$, that is, show that two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal if and only if

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

Exercise* 2. (3 marks) Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{y} = (1, 1, -1)^T$.

(a) Find a basis of $S^\perp$. [Hint: $S^\perp$ is the nullspace of a 1×3 matrix.]

(b) Give a geometric description of S and $S^\perp$.

Exercise* 3. (2 marks) Let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be vectors in $\mathbb{R}^n$ and let $S = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_r)$. Show that $\mathbf{x} \in S^\perp$ if and only if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \dots, r$.

Exercise 4. Let S be a subspace of $\mathbb{R}^n$. Show the following:

(a) $S^\perp$ is a subspace of $\mathbb{R}^n$.

(b) $(S^\perp)^\perp = S$.

Exercise* 5. (2 marks) Let A be an arbitrary matrix, and let $\mathbf{x}$ be in the column space of A . If $A^T \mathbf{x} = \mathbf{0}$ what is $\mathbf{x}$? [Hint: Use the Fundamental Subspace Theorem.]

Exercise* 6. (4 marks) Let $\mathbf{y} = (3, 4)^T$ and $\mathbf{u} = (2, 1)^T$. Let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Find $\hat{\mathbf{y}}$. Calculate $\mathbf{y} - \hat{\mathbf{y}}$ and verify that $\mathbf{u}$ and $\mathbf{y} - \hat{\mathbf{y}}$ are orthogonal. Give a geometric description of the quantity $\|\mathbf{y} - \hat{\mathbf{y}}\|$.

Exercise* 7. (3 marks) Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Show the following:

(a) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;

(b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$;

(c) $U\mathbf{x}$ and $U\mathbf{y}$ are orthogonal if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

[Hint for (a): use the fact that $U^T U = I$.]

Exercise* 8. (4 marks) Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be the vectors in $\mathbb{R}^3$ given by

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}.$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express $\mathbf{y} = (3, 7, 11)^T$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

(c) Determine an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ such that $\text{Span}(\mathbf{v}_1) = \text{Span}(\mathbf{u}_1)$ and $\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \text{Span}(\mathbf{u}_1, \mathbf{u}_2)$.